Shapes of river networks and leaves: are they statistically similar?

Jon D. Pelletier and Donald L. Turcotte

Email alerting service	Receive free email alerts when new articles cite this article - sign up in the box at the top right-hand corner of the article or click **here**

THE ROYAL
SOCIETY

Shapes of river networks and leaves: are they statistically similar?

Jon D. Pelletier[1]* and Donald L. Turcotte[2]

[1]*Division of Geological and Planetary Science, Mail Stop 150-21, California Institute of Technology, Pasadena, CA 91125, USA*
[2]*Department of Geological Sciences, Snee Hall, Cornell University, Ithaca, NY 14853, USA*

The structure of river networks is compared with the vein structure of leaves. The two structures are visually similar at the smaller scales. The statistics of branching and side branching are nearly identical. The branching structure of diffusion-limited aggregation clusters is also similar and can provide an explanation for the structure of river networks. The origin of the self-similar branching and side branching of the vein structure in leaves is not clear but it appears to be an optimal network in terms of transporting nutrients to all parts of the leaf with the least total resistance.

Keywords: leaf veins; fractal; network; Tokunaga; rivers; optimal

Fractal trees have been associated with a wide variety of naturally occurring networks. Examples include river networks, actual plants and trees, root systems, bronchial systems and cardiovascular systems. Before the introduction of fractals by Mandelbrot (1967), empirical studies of drainage networks (Horton 1945) had given power-law relations between stream numbers, stream lengths, drainage areas and stream slopes.

Peckham (1995) developed a software routine for obtaining river networks from digital elevation models (DEMs). Seven typical drainage networks obtained using this routine are given in figure 1. The routine also gives a variety of data on networks, including numbers and lengths of streams and drainage areas. We have also used this routine to obtain the vein structures of leaves. Two examples are given in figure 2. The vein structure of a whole mature leaf and a side lobe of a leaf of *Sorbus hybrida* (Rosaceae) were obtained from greyscale images given by Merrill (1978). The leaf architecture of the whole leaf has a well-defined regularity at the largest scale but becomes quite random at smaller scales as shown by the vein structure of the side lobe. The small-scale structure of the leaf strongly resembles the river networks in figure 1.

To compare leaves with drainage networks it is necessary to quantify the branching structures. The original branch-ordering taxonomy for fractal trees was developed as a stream-ordering system in geomorphology by Horton (1945) and Strahler (1957). Streams on a standard topographic map with no upstream tributaries are defined to be first order $(i = 1)$. When two first-order streams combine they form a second-order $(i = 2)$ stream. When two second-order streams combine, they form a third-order $(i = 3)$ stream, and so forth. Horton (1945) also introduced the bifurcation ratio

$$R_b = \frac{N_i}{N_{i+1}}, \tag{1}$$

and the length-order ratio

$$R_r = \frac{r_{i+1}}{r_i}, \tag{2}$$

where N_i is the number of streams of order i and r_i is the mean length of streams of order i. Emprically it was found that R_b and R_r were nearly constant, independent of order, for actual drainage networks.

With the introduction of the fractal dimension D as the power-law scaling exponent between number and length, it was recognized that the fractal dimension of a stream network is given by

$$D = \frac{\ln R_b}{\ln R_r}. \tag{3}$$

If a self-similar network is area filling we must have $D = 2$.

An important aspect of the networks illustrated in figures 1 and 2 is side branching. That is, some first-order streams intersect second-order, third-order, and all higher-order streams. Similarly, second-order streams intersect third-order and higher-order streams, and so forth. To classify side branching Tokunaga (1978) extended the Strahler (1957) ordering system. A first-order branch intersecting a first-order branch is denoted '11' and the number of these branches is N_{11}, a first-order branch intersecting a second-order branch is denoted '12' and the number of such side branches is N_{12}, a second-order branch intersecting a second-order branch is denoted '22' and the number of such branches is N_{22}, and so forth. The total number of streams of order i, N_i, is related to N_{ij} by

$$N_i = \sum_{j=1}^{n} N_{ij}, \tag{4}$$

for a fractal tree of order *n*. The branch numbers N_{ij}, $i < j$, constitute a square upper-triangular matrix. This class of fractal trees can also be quantified in terms of branching ratios T_{ij}. These are the average number of branches of
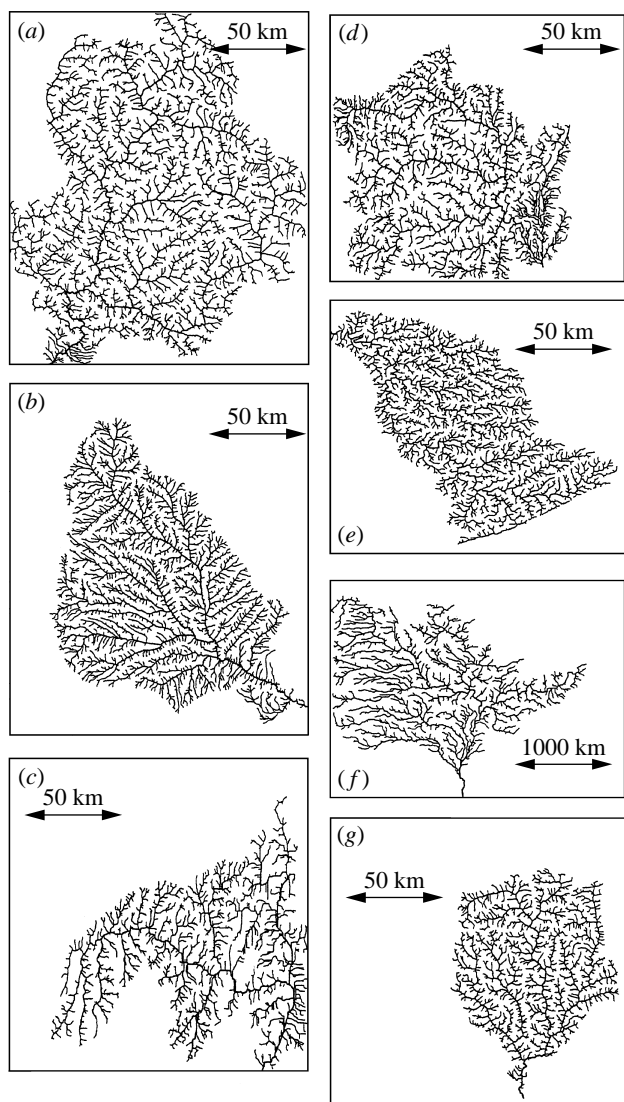
Figure 1. Drainage networks analysed: (*a*) Kumaun;
(*b*) Loess Plateau, Shanxi Province, China; (*c*) Schoharie
Creek, New York; (*d*) Nepal; (*e*) Kentucky River;
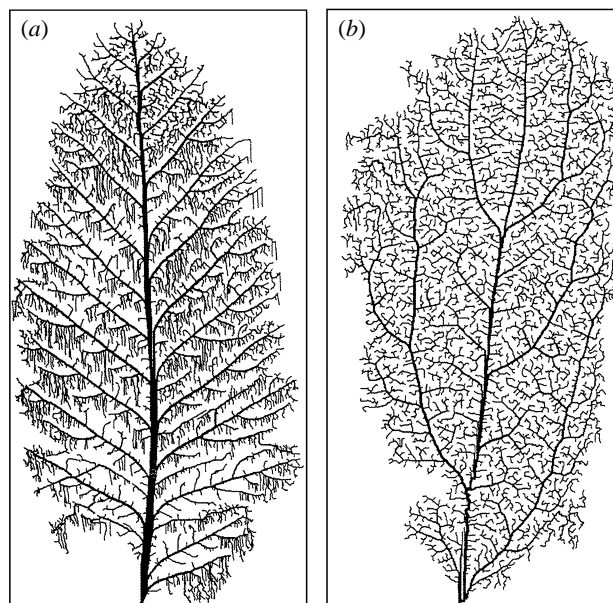(*f*) Mississippi River; and (*g*) Bhutan.

Figure 2. Vein structure of a mature leaf in *Sorbus* L.
(Rosaceae) obtained by using a scanning routine on greyscale
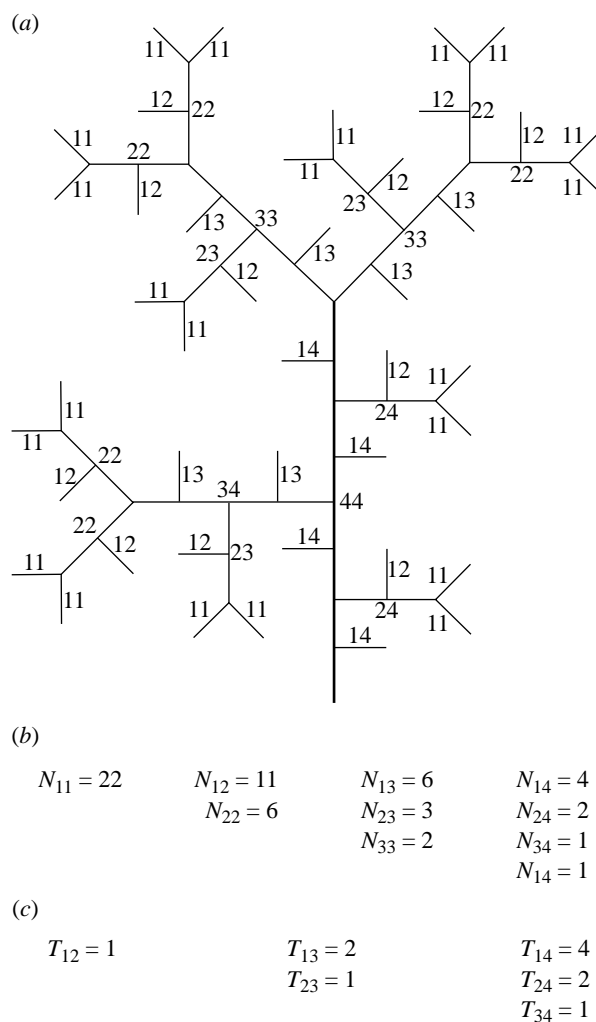images given by Merrill (1978): (*a*) whole leaf; (*b*) side lobe.

order i joining branches of order j. Branching ratios are
related to branch numbers by

$$T_{ij} = \frac{\mathcal{N}_{ij}}{\mathcal{N}_j}. \tag{5}$$

Again the branching ratios T_{ij} constitute a square, upper-
triangular matrix.

We now define self-similar trees to be the subset of trees
for which $T_{i,i+k} = T_k$ where T_k is a branching ratio that
depends on k but not on i. Tokunaga (1978) introduced a
more restricted class of self-similar, side-branching trees
by requiring for self-similarity of side branching that

$$T_k = ac^{k-1}. \tag{6}$$

This is now a two-parameter family of trees and we will
define fractal trees in this class to be Tokunaga trees.

A fourth-order Tokunaga tree is illustrated in figure 3*a*.
The corresponding branch-number and branching-order
matrices are given in figure 3*b*,*c*. For this construction we
have $R_r = 2$ and $\mathcal{N}_4 = 1$, $\mathcal{N}_3 = 3$, $\mathcal{N}_2 = 11$, and $\mathcal{N}_1 = 43$.

(*b*)

$N_{11} = 22$	$N_{12} = 11$	$N_{13} = 6$	$N_{14} = 4$
	$N_{22} = 6$	$N_{23} = 3$	$N_{24} = 2$
		$N_{33} = 2$	$N_{34} = 1$
			$N_{14} = 1$

(*c*)

$T_{12} = 1$	$T_{13} = 2$	$T_{14} = 4$
	$T_{23} = 1$	$T_{24} = 2$
		$T_{34} = 1$

Figure 3. (*a*) Fourth-order Tokunaga fractal tree. (*b*) Branch-
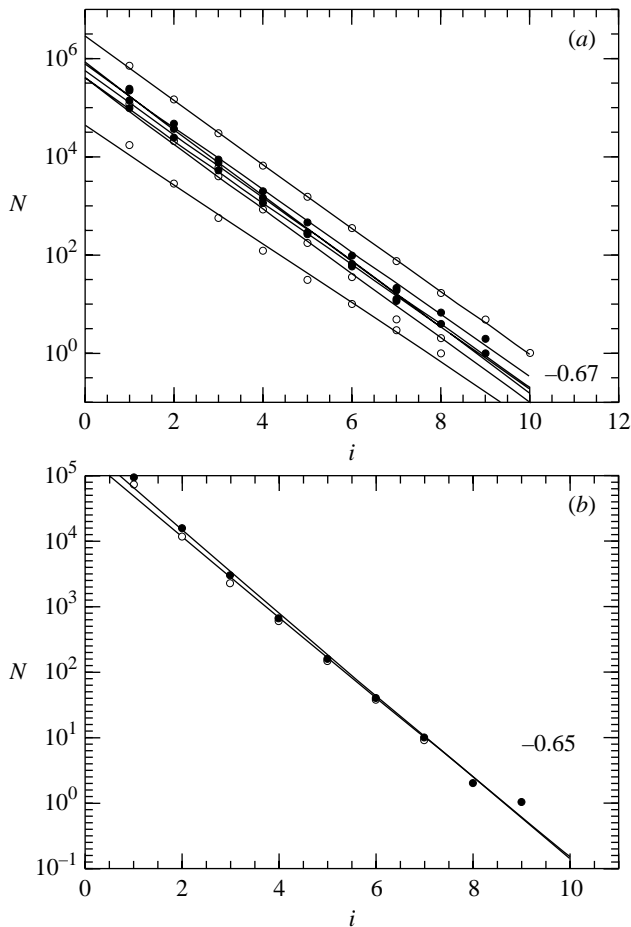number matrix. (*c*) Branching-ratio matrix.

Figure 4. Plot of the average number of branches of order i as a function of order i. (a) Results for the seven drainage basins given in figure 1. The data correlate with $\mathcal{N} \propto i^{-0.67}$; thus $R_b = 10^{0.67} = 4.68$. (b) Results for the two leaf networks given in figure 2. The data correlate with $\mathcal{N} \propto i^{-0.65}$; thus $R_b = 10^{0.65} = 4.47$.

Figure 5. Plot of the average branch length of order i as a function of order i. (a) Results for the seven drainage basins given in figure 1. The data correlate with $r \propto i^{-0.35}$; thus $R_b = 10^{0.35} = 2.24$. (b) Results for the two leaf networks given in figure 2. The data correlate with $r \propto i^{-0.34}$; thus $R_b = 10^{0.34} = 2.19$.

The bifurcation ratio R_b is not constant but approaches 4 for large i-values. Thus from equation (3), D approaches 2 for large values of i. We also find that $T_{12} = T_{23} = T_{34} = T_1 = 1$, $T_{13} = T_{24} = 2$ and $T_{14} = 4$, so that from equation (6) we have $a = 1$ and $b = 2$.

We now quantify the branching networks illustrated in figures 1 and 2. Peckham (1995) has determined branching-ratio matrices for the Kentucky River basin in Kentucky and the Powder River basin in Wyoming. Both are eighth-order basins with the Kentucky River basin having an area of 13 500 km^2 and the Powder River basin an area of 20 181 km^2. For the Kentucky River basin $R_b = 4.6$ and $R_r = 2.5$; for the Powder River basin $R_b = 4.7$ and $R_r = 2.4$. From equation (3) the corresponding values of the fractal dimension are $D = 1.67$ and $D = 1.77$, respectively. Good agreement with the Tokunaga relation (6) was obtained, taking $a = 1.2$ and $c = 2.5$.

For the river networks illustrated in figure 1 the number-order statistics are given in figure 4a. We find $R_b = 10^{0.67} = 4.68$. The length-order statistics are given in figure 5a. We find $R_r = 10^{0.35} = 2.24$. Thus from equation (3) we have $D = 1.91$. The mean values of the branching ratios T_k are given in figure 6a. Quite good agreement with the Tokunaga relation (6) is obtained, taking $a = 1.58$ and $c = 2.44$. In addition, we have
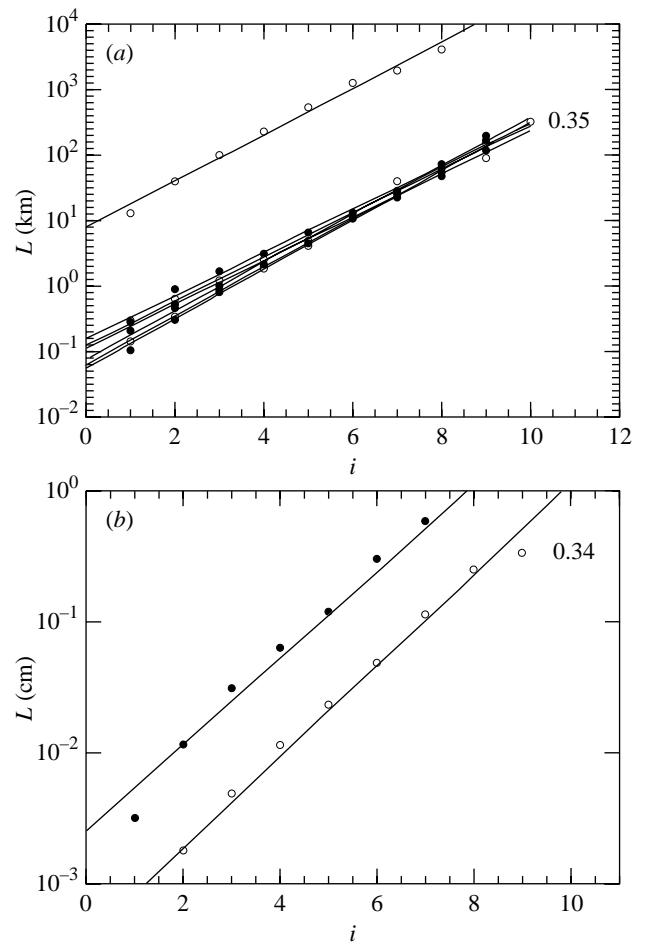
plotted the relationship between length and area averaged for each Strahler order in figure 7a. The results indicate that $L \propto A^q$ with q nearly equal to one-half. This relationship is known as the Hack law (Hack 1957).

For the leaves illustrated in figure 2 the number-order statistics are given in figure 4b; we find that $R_b = 10^{0.65} = 4.47$. The length-order statistics are given in figure 5b; we find that $R_r = 10^{0.34} = 2.19$. Thus from equation (3) we have $D = 1.91$. The mean values of the branching ratios T_k are given in figure 6b. Again quite good agreement with the Tokunaga relation (6) is obtained, taking $a = 1.23$ and $c = 2.34$. The same relationship between length and area observed for drainage networks is observed for leaf veins (figure 7b).

At larger scales (large k) where there is regularity in the side branching, the statistics of side branching deviate from the Tokunaga relation (6). Both in terms of number-length statistics and in terms of side-branching statistics we find that drainage networks and the vein structure of leaves at the smaller scales are very similar. This confirms the visual similarities between figures 1 and 2. An obvious question is, why do drainage networks and leaves develop a Tokunaga branching structure? We first note the similarities with diffusion-limited aggregation (DLA).
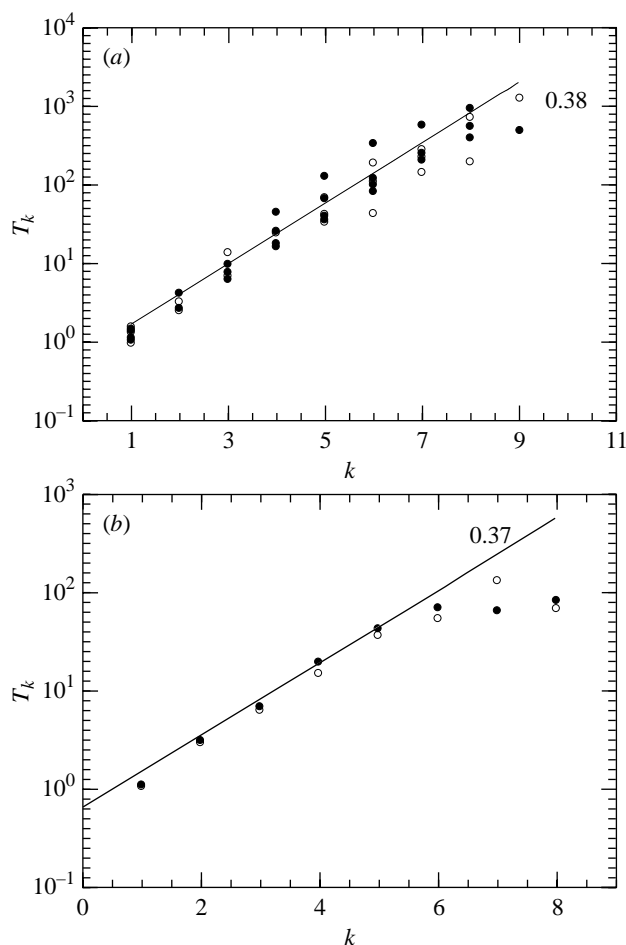
Figure 6. Plot of the Tokunaga branching ratios T_k as a function of k. (*a*) Results for the seven drainage networks illustrated in figure 1. The data correlate with equation (6), taking $a = 1.58$ and $c = 2.44$. (b) Results for the two leaves illustrated in figure 2. The data at lower order correlate with equation (6), taking $a = 1.23$ and $c = 2.34$.

Figure 7. Plot of the length versus drainage area averaged for each Strahler order i. (*a*) Results for the seven drainage basins given in figure 1. The data correlate with $L \propto A^{0.5}$. (b) Results for the two leaf networks given in figure 2. The data correlate with $L \propto A^{0.5}$ also.

The concept of DLA was introduced by Witten & Sander (1981). They considered a grid of points on a two-dimensional lattice and placed a seed particle near the centre of the grid. An accreting particle was randomly introduced on a 'launching' circle and was allowed to follow a random path until (i) it accreted to the growing cluster of particles by entering a grid point adjacent to the cluster, or (ii) it wandered across a larger 'killing' circle. The resulting sparse, tree-like structure has been taken as an excellent representation of dendritic growth patterns found both in nature and in industrial applications (Vannimenus & Viennot 1989).

Ossadnik (1992) has considered the branching statistics of 47 off-lattice DLA clusters each with 10^6 particles. On average, the networks were 11th-order fractal trees. The average bifurcation ratio for the clusters was found to be $R_b = 5.15 \pm 0.05$ and the average length-order ratio $R_r = 2.86 \pm 0.05$; from equation (3) the corresponding fractal dimension is $D = 1.56$. The DLA clusters are much less dense than the drainage networks or leaf vein networks, and thus the fractal dimension for DLA clusters are considerably smaller. In order to analyse the branching statistics of DLA clusters, Ossadnik (1992) used the ramification matrix introduced for DLA by Vannimenus & Viennot
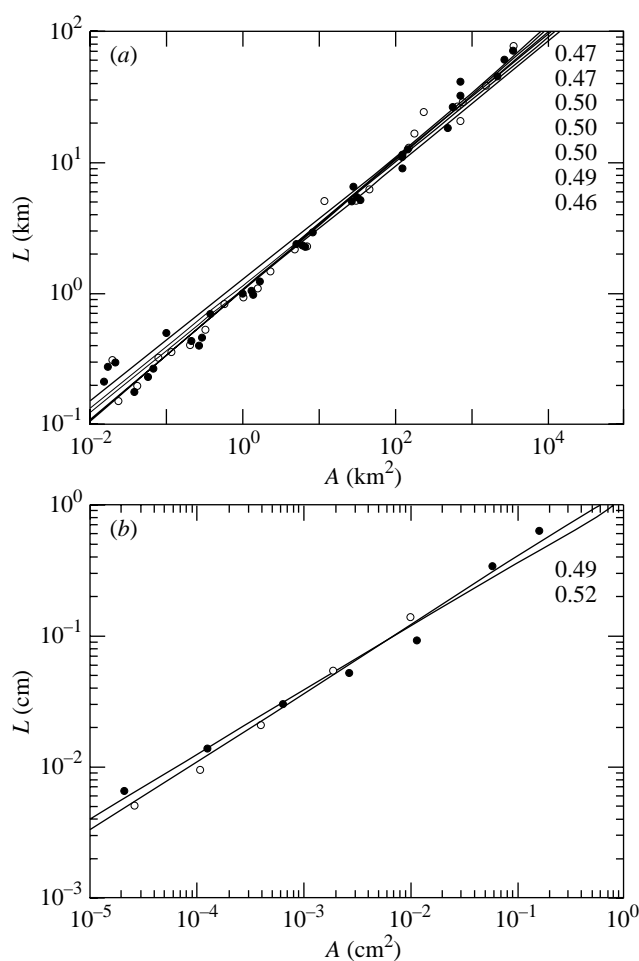
(1989). The ramification matrix is entirely equivalent to the branching-ratio matrix introduced by Tokunaga (1978). In terms of the Tokunaga trees defined in equation (6) it is found that $a = 1.5$ and $c = 2.7$ for DLA trees. Thus DLA clusters, river networks and leaves all exhibit Tokunaga side-branching statistics.

A wide variety of models have been proposed for drainage networks. These have been reviewed by Rodriguez-Iturbe & Rinaldo (1997) and by Turcotte (1997). Masek & Turcotte (1993) proposed a model for headward migration of drainage networks based on DLA. This model yields Tokunaga statistics which are in good agreement with equation (6). Peckham (1995) has shown that there are large discrepancies between river network statistics and random growth networks such as that proposed by Shreve (1969).

Some authors (Rodriguez-Iturbe & Rinaldo 1997) have shown that river networks are optimal networks for transporting run-off with the minimum stream power exerted on the landscape. Because the structures of leaves and river basins are statistically similar we can conclude that evolution has found an optimal structure for transporting nutrients to and from cells in the leaf by the trial and error process of mutation and natural selection.

REFERENCES

Fowler, A. D. 1990 Self-organized mineral textures of igneous rocks—the fractal approach. *Earth Sci. Rev.* **29**, 47–55.

Hack, J. T. 1957 Studies of longitudinal profiles in Virginia and Maryland. *US Geol. Surv. Prof. Pap.* 294B.

Horton, R. E. 1945 Erosional development of streams and their drainage basins; hydrophysical approach to quantitative geomorphology. *Bull. Geol. Soc. Am.* **56**, 275–370.

Mandelbrot, B. B. 1967 How long is the coastline of Britain? Statistical self-similarity and the fractal dimension. *Science* **156**, 636–638.

Masek, J. G. & Turcotte, D. L. 1993 A diffusion-limited aggregation model for the evolution of drainage networks. *Earth Planet. Sci. Lett.* **119**, 379–386.

Merrill, E. K. 1978 Comparison of mature leaf architecture of three types in *Sorbus* L. (Rosaceae). *Bot. Gaz.* **139**, 447–453.

Ossadnik, P. 1992 Branch order and ramification analysis of large diffusion-limited aggregation clusters. *Phys. Rev.* A **45**, 1058–1066.

Peckham, S. 1995 New results for self-similar trees with applications to river networks. *Water Resour. Res.* **31**, 1023–1029.

Rodriguez-Iturbe, I. & Rinaldo, A. 1997 *Fractal river basins*. New York: Cambridge University Press.

Shreve, R. L. 1969 Stream lengths and basin areas in topologically random channel networks. *J. Geol.* **77**, 397–405.

Strahler, A. N. 1957 Quantitative analysis of watershed geomorphology. *Am. Geophys. Union Trans.* **38**, 913–920.

Tokunaga, E. 1978 Consideration on the composition of drainage networks and their evolution. *Geogr. Rep. Tokyo Metro. Univ.* **13**, 1–27.

Turcotte, D. L. 1997 *Fractals and chaos in geology and geophysics*. New York: Cambridge University Press.

Vannimenus, J. & Viennot, X. G. 1989 Combinatorial tools for the analysis of ramified patterns. *J. Statist. Phys.* **54**, 1529–1538.

Witten, T. A. & Sander, L. M. 1981 Diffusion-limited aggregation, a kinetic critical phenomenon. *Phys. Rev. Lett.* **47**, 1400–1403.